Lesson Plan for Teachers

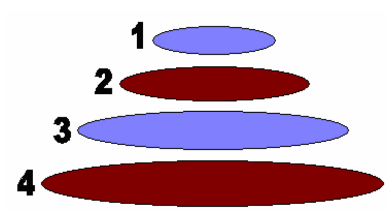
Grade level recommendation: 8^{th} grade

Learning goals:
- Problem solving
- Reasoning
- Basic algebra
  - Exponents
  - Recursive equations
  - Explicit equations

NCTM standards correlation: http://www.nctm.org/standards/

8th Grade Number and Operations: "Students use exponents and scientific notation to describe very large and very small numbers."

Materials per group:
- Scientific calculator
- Four disks of differing sizes and alternating colors
  - gold and silver colored disks are recommended (see legend on student activity sheet)
- Sheet of paper with labeled locations (see diagram below).



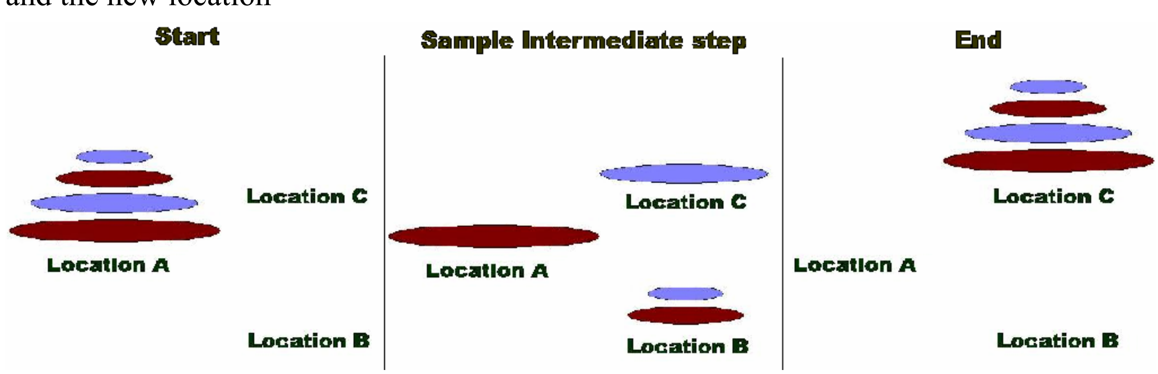
Preparation:

Divide students into groups of 2-4 participants

Directions for playing the game:

Objective: move all of the disks from location A to location C (see diagram below)

Rules:
1. A larger disk can never be placed on top of a smaller disk
2. One disk may be moved at a time and the disk must be the top disk at both the old and the new location



Student Activity Sheet **ANSWERS**

The Legend:

In an ancient city in India, so the legend goes, monks in a temple have to move a pile of 64 gold and silver disks from one location to another.

The disks are heavy and fragile; only one can be carried at a time and a larger disk may not be placed on top of a smaller disk. The monks started moving disks back and forth, between the original pile, the pile at the new location, and the intermediate location, always keeping the piles in order (largest on the bottom, smallest on the top).

It is said that as soon as the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder. If the monks move one disk every second, how long will take for the world to end?

Rules:

- Only one disk can be moved at a time
- A larger disk may not be placed on top of a smaller disk
- There are only three locations where you may place the disks
- The objective is to move all the disks to a new location.

Getting started:

What is the minimum number of moves it takes to move 2 disks? 3 disks? 4 disks? Fill in your findings below.

Number of disks	Minimum number of moves
1	1
2	3
3	7
4	15

Having trouble deciding where to move the next disk? (Hint: Do certain colors always move in a certain direction?)

In an optimal (least number of moves) solution, all the silver disks always move in a one direction (clockwise or counter-clockwise), while all the gold disks move in the opposite direction. In this case, disks 1 & 3 always move in a clockwise direction while disks 2 & 4 move in a counter-clockwise direction.

Check your answers with other groups. Does everyone agree on the minimum number of moves?

Please make sure all students have the correct answers as it will be difficult to derive a recursive equation from incorrect data.

Can you see a pattern yet? Do you think you need more numbers?

Students are not expected to discern a pattern yet.

Getting tired of physically moving the disks and counting the number of turns? How else can you calculate the minimum number of moves for each tower? Can you find a mathematical pattern to help you find the number of moves to move a pile of 64 disks?

Students should agree that trying to solve the legend (64 disks) by hand would be time consuming and impractical. The discussion should then touch on different ways to approach the problem mathematically, with a focus on recursive and explicit equations. It should be mentioned that recursive patterns can help generate enough data for an explicit (non-recursive) equation to be derived.

Finding a recursive pattern (a pattern that uses information from one step to find the next step):
We now know how many turns it takes to move an n-disk tower (of up to four disks). Let's use that to find how many turns it takes to move an (n+1)-disk tower without having to move all those disks!

Prompt students to focus on patterns:
Q: What must happen before the bottom disk can be moved to location C?
A: All smaller disks must be on location B

Working with only **3** (n) disks answer the following:

1. What is the minimum number of moves it takes to move the two smaller disks from the original location to the intermediate location?
   3 [Let's call this M]
2. How many moves does it now take to move the largest disk to the final location?
   1
3. How many moves does it take to move the two smaller disks from the intermediate location to the final location?
   3
4. What is the total number of moves? Does it coincide with what you found before?
   7 [Let's call this M_n]

What do you notice about your answers for questions 1&3?

These answers require the same number of moves (i.e. 3 moves).

Use your answers to questions 1-3 to find a generalized equation (using M) that will give you the answer to question 4.

M_n = 2M+1

Now do the same using **4** disks (that is, n+1 disks).

5. What is the minimum number of moves it takes to move the three smallest disks from the original location to the intermediate location?
   7

6. How many moves does it now take to move the largest disk to the final location?
   1
7. How many moves does it take to move the three smallest disks from the intermediate location to the final location?
   7
8. What is the total number of moves? Does it coincide with what you found before?
   15

Does the generalized equation you found earlier still work?

It should.
In this case: $M_n = 2M+1$ → $M_n = 2(7)+1$ → $M_n = 15$

What do you notice about the total number of turns required to move three disks (answer to question 4) and your answer to question 5? Might this be a recursive pattern?

These numbers are the same. A recursive equation can be derived: if $M(n)$ is the minimum number of turns it takes to move n disks from one post to another, then $M(n+1) = 2\,M(n) + 1$.

Use the formula you found to calculate the minimum number of moves for a few more towers:

Number of disks (n)	2(M)+1	Minimum number of moves (M_n)
1		1
2	2 (**1**) + 1	3
3	2 (**3**) + 1	7
4	2 (7) + 1	15
5	2 (15) + 1	31
6	2 (31) + 1	63
7	2 (63) + 1	127

Notice that in order to use this recursive equation, you would always have to know the minimum number of moves (M_n) of the preceding (one disk smaller) tower. So, to find the number of moves it would take to transfer 64 disks to a new location, we would also have to know the number of moves for a 63-disk tower, a 62-disk tower, a 61-disk tower, and so on.

Another recursive pattern students might notice is that to get the next minimum number of moves, you add the next power of 2 to the previous minimum number of moves.

Let's try to find an explicit (non-recursive) equation:
Do you see a pattern in the M_n s you calculated? Does this pattern look familiar?
[Hint: Does it look more familiar if you add 1 to each M_n?]

Students should notice that the M_n is always one unit less than 2 to the power of *n*.

Turn this pattern into an equation: 2^n-1

Back to the Legend:
If the monks move one disk every second, how many years will it take them to finish moving all 64 disks? Are you surprised by your answer?

2^(64) – 1 → 18,446,744,073,709,551,615 seconds → ~584.542 billion years

Wrap-up debrief questions and/or teaching points

What are recursive equations?

"Recursive formula is a formula that is used to determine the next term of a sequence using one or more of the preceding terms." Taken from http://www.icoachmath.com/SiteMap/RecursiveFormula.html

In what situations are recursive equations useful?

When all preceding terms are known

What are the limitations of recursive equations?

When not all the preceding terms are available
When trying to solve for a large n^{th} term, in our case 64

How does an explicit equation differ from a recursive equation?

An explicit equation does not reference itself

Did students expect the legend to produce such a large number?

Discuss exponential equations and why numbers get so big so quickly.

Hanoi Towers: Student Activity Sheet

The Legend:

In an ancient city in India, so the legend goes, monks in a temple have to move a pile of 64 gold and silver disks from one location to another.

The disks are heavy and fragile; only one can be carried at a time and a larger disk may not be placed on top of a smaller disk. The monks started moving disks back and forth, between the original pile, the pile at the new location, and the intermediate location, always keeping the piles in order (largest on the bottom, smallest on the top).

It is said that as soon as the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder. If the monks move one disk every second, how years will take for the world to end?

Rules:

- Only one disk can be moved at a time
- A larger disk may not be placed on top of a smaller disk
- There are only three locations where you may place the disks
- The objective is to move all the disks to a new location.

Getting started:

What is the minimum number of moves it takes to move 2 disks? 3 disks? 4 disks? Fill in your findings below.

Number of disks	Minimum number of moves
1	1
2	
3	
4	

Having trouble deciding where to move the next disk? (Hint: Do certain colors always move in a certain direction?)

Check your answers with other groups. Does everyone agree on the minimum number of moves?

Can you see a pattern yet? Do you think you need more numbers?

Getting tired of physically moving the disks and counting the number of turns? How else can you calculate the minimum number of moves for each tower? Can you find a mathematical pattern?

Finding a recursive pattern (a pattern that uses information from one step to find the next step):
We now know how many turns it takes to move an *n*-disk tower (of up to four disks). Let's use that to find how many turns it takes to move an (*n*+1)-disk tower without having to move all those disks!

Working with only **3** (*n*) disks answer the following:

9. What is the minimum number of moves it takes to move the two smaller disks from the original location to the intermediate location?
   _________ [Let's call this *M*]
10. How many moves does it now take to move the largest disk to the final location?
   ____
11. How many moves does it take to move the two smaller disks from the intermediate location to the final location?
   ____
12. What is the total number of moves? Does it coincide with what you found before?
   _________ [Let's call this M_n]

What do you notice about your answers for questions 1&3?

Use your answers to questions 1-3 to find a generalized equation (using *M*) that will give you the answer to question 4.

M_n =

Now do the same using **4** disks (that is, *n*+1 disks).

13. What is the minimum number of moves it takes to move the three smallest disks from the original location to the intermediate location?
   ____
14. How many moves does it now take to move the largest disk to the final location?
   ____
15. How many moves does it take to move the three smallest disks from the intermediate location to the final location?
   ____
16. What is the total number of moves? Does it coincide with what you found before?
   ____

Does the generalized equation you found earlier still work?

What do you notice about the total number of turns required to move three disks (answer to question 4) and your answer to question 5? Might this be a recursive pattern?

Use the formula you found to calculate the minimum number of moves for a few more towers:

Number of disks (n)	2(M)+1	Minimum number of moves (M_n)
1		1
2	2 (**1**) + 1	3
3	2 (**3**) + 1	7
4	2 (___) + 1	
5	2 (___) + 1	
6	2 (___) + 1	
7	2 (___) + 1	

Notice that in order to use this recursive equation, you would always have to know the minimum number of moves (M_n) the preceding (one disk smaller) tower. So, to find the number of moves it would take to transfer 64 disks to a new location, we would also have to know the number of moves for a 63-disk tower, a 62-disk tower, a 61-disk tower, and so on.

Let's try to find an explicit (non-recursive) equation:
Do you see a pattern in the M_n s you calculated? Does this pattern look familiar?
[Hint: Does it look more familiar if you add 1 to each M_n?]

Turn this pattern into an equation: ____________________

Back to the Legend:
If the monks move one disk every second, how many years will it take them to finish moving all 64 disks? Are you surprised by your answer?